

10 Week 10 - Fourier transform optics.

10.1 Background

We highlight the derivation of the relation between an object at the focal plane of a lens ($z = -f$) and the Fourier transform of the object at the back focal plane ($z = f$).

10.2 The free space propagator from a point source

The wave equation for a scalar field, such as the free-space components of \vec{E} and \vec{B} , designated $\psi(\vec{r}, t)$, is

$$\nabla^2 \psi(\vec{r}, t) - c^2 \frac{\partial^2}{\partial t^2} \psi(\vec{r}, t) = 0 \quad (10.1)$$

where c is the speed of light in vacuum. If we take

$$\psi(\vec{r}, t) = u(\vec{r})e^{i\omega t} \quad (10.2)$$

and designate $k = c/\omega$, we have

$$\nabla^2 u(\vec{r}) + k^2 u(\vec{r}) = 0 \quad (10.3)$$

In spherical coordinates, the field emanating from a point is given by

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} u(r) + k^2 u(r) = -\delta(\vec{r}). \quad (10.4)$$

This is solved by

$$u(r) = \frac{e^{\pm ikr}}{4\pi r} \quad (10.5)$$

where the sign is the direction of propagation. We use plus so that the field propagates outwards radially with an amplitude that diminishes with distance and an evolving phase.

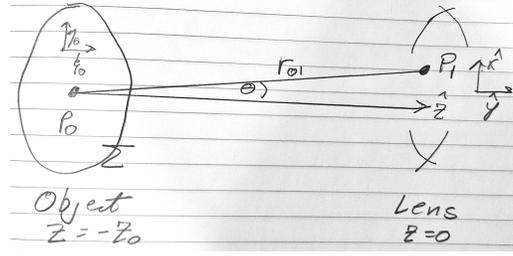
10.3 The free space propagator from an extended source

Here we jump to the derivation of the scalar field propagator derived by Rayleigh and Sommerfeld, a correction of earlier work by Kirchoff, to calculate the scalar field at a point P_1 from a point P_0 that is a distance r_{01} away. The underlying idea is based on the linear propagation of light in free space. We label the optical axis by z (see Figure). The coordinates at P_0 are (ζ, η) while those at P_1 are (x, y) . The axis of propagation is lifted from the optical axis by an angle θ . Rayleigh and Sommerfeld write

$$U_{P_1}(x, y, z) = \frac{k}{i2\pi} \int \int_{\Sigma} d\zeta d\eta U_{P_0}(\zeta, \eta, z) \frac{e^{ikr_{01}}}{r_{01}} \cos \theta \left(1 - \frac{1}{ikr_{01}}\right) \quad (10.6)$$

where $k_x^2 + k_y^2 + k_z^2 = \vec{k}^2$.

Figure 1: from Goodman



10.3.1 Paraxial approximation

We now simplify for the case of small θ , in what is called the paraxial or far-field approximation. Here

$$k_x^2 + k_y^2 \ll k_z^2 \quad (10.7)$$

and

$$kr_{01} \gg 1. \quad (10.8)$$

Noting that

$$r_{01}^2 = z^2 + (x - \zeta)^2 + (y - \eta)^2 \quad (10.9)$$

and that the deviations from $r_{01} = z$ are small, we have the zero-order result that is valid for the magnitude

$$r_{01} = z \quad (10.10)$$

so that

$$\begin{aligned} \cos \theta &\equiv \frac{z}{r_{01}} \\ &\simeq 1 \end{aligned} \quad (10.11)$$

and the first-order result that is valid for the phase

$$\begin{aligned} r_{01} &= z \sqrt{1 + \left(\frac{x - \zeta}{z}\right)^2 + \left(\frac{y - \eta}{z}\right)^2} \\ &\simeq z \left[1 + \frac{1}{2} \left(\frac{x - \zeta}{z}\right)^2 + \frac{1}{2} \left(\frac{y - \eta}{z}\right)^2 \right] \\ &\simeq z + \frac{1}{2z} [(x - \zeta)^2 + (y - \eta)^2]. \end{aligned} \quad (10.12)$$

Thus

$$U_{P_1}(x, y, z) = \frac{k}{i2\pi z} e^{ikz} \iint_{\Sigma} d\zeta d\eta U_{P_0}(\zeta, \eta, z) e^{\frac{ik}{2z} [(x - \zeta)^2 + (y - \eta)^2]}. \quad (10.13)$$

This reduction is referred to as the Fresnel propagator.

10.3.2 An aside

The Fresnel propagator is in the form of a convolution, i.e.,

$$U_{P_1}(x, y, z) = \frac{k}{i2\pi z} e^{ikz} \int \int_{\Sigma} d\zeta d\eta U_{P_0}(\zeta, \eta, z) h(x - \zeta, y - \eta). \quad (10.14)$$

More so, when we rewrite the kernel h as

$$e^{\frac{ik}{2z}[(x-\zeta)^2+(y-\eta)^2]} = e^{\frac{ik(x^2+y^2)}{2z}} e^{\frac{ik(\zeta^2+\eta^2)}{2z}} e^{\frac{ik(\zeta x+\eta y)}{z}}. \quad (10.15)$$

we see that U_{P_1} is the Fourier transform of U_{P_0} times a quadratic phase factor, i.e.,

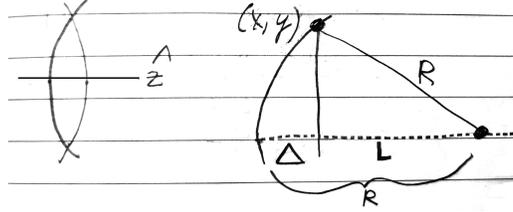
$$U_{P_1}(x, y, z) = \frac{k}{i2\pi z} e^{ikz} e^{\frac{ik(x^2+y^2)}{2z}} \int \int_{\Sigma} d\zeta d\eta \left[U_{P_0}(\zeta, \eta, z) e^{\frac{ik(\zeta^2+\eta^2)}{2z}} \right] e^{\frac{ik(\zeta x+\eta y)}{z}}. \quad (10.16)$$

We shall seek a solution that avoids the extra phase, which is a distortion.

10.4 The effect of a lens

The last bit of preliminary material is to understand how a lens affects the phase of light. This determination follows the analysis of the bending of light when the incident angles are small. We consider a biconvex lens where each side has a radius of curvature R (see Figure). We are interested in the distance through the glass as a function of x and y away from the center. With reference to the Figure, for one side of the lens

Figure 2: from Goodman



$$\begin{aligned} \Delta &\equiv R - L & (10.17) \\ &= R - R\sqrt{R^2 - x^2 - y^2} \\ &= R - \left(1 - \sqrt{1 - \frac{x^2 + y^2}{R^2}}\right) \\ &= R - \left(1 - 1 - \frac{x^2 + y^2}{2R^2} + \dots\right) \\ &\simeq \frac{x^2 + y^2}{2R}. \end{aligned}$$

This leads to a retarding phase shift by a biconvex lens, denoted $T(x, y)$, or

$$\begin{aligned} T(x, y) &= e^{-ik(n-1)(x^2+y^2)2\Delta} & (10.18) \\ &= e^{\frac{-ik(x^2+y^2)}{2f}} \end{aligned}$$

where f is the focal length with $f = R/2(n - 1)$ and n is the index of the glass.

10.5 Fresnel propagation through a lens

The first step is to propagate from an object plane at $z = -z_o$ to just before a lens at $z = 0$, as shown in the Figure

$$U_l(x, y, 0^-) = \frac{k}{i2\pi z_o} e^{ikz_o} \int \int_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) e^{\frac{ik}{2z_o} [(x-\zeta_o)^2 + (y-\eta_o)^2]}. \quad (10.19)$$

The second step is to multiply by $T(x, y)$ to propagate through the lens.

$$U_l(x, y, 0^+) = \frac{k}{i2\pi z_o} e^{ikz_o} e^{\frac{-ik(x^2+y^2)}{2f}} \int \int_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) e^{\frac{ik}{2z_o} [(x-\zeta_o)^2 + (y-\eta_o)^2]}. \quad (10.20)$$

The third and last step is to use a Fresnel propagator to go from the lens to the image plane at $z = z_i$, i.e.,

$$\begin{aligned} U_i(x, y, z_i) &= \frac{k}{i2\pi z_i} e^{ikz_i} \int \int_{\Sigma_l} d\zeta_l d\eta_l U_l(\zeta_l, \eta_l, 0^+) e^{\frac{ik}{2z_i} [(x-\zeta_l)^2 + (y-\eta_l)^2]} \\ &= \frac{k^2}{(i2\pi)^2 z_o z_i} e^{ik(z_o+z_i)} \int \int_{\Sigma_l} \int \int_{\Sigma_o} d\zeta_l d\eta_l d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) \\ &\quad \times e^{\frac{-ik}{2f} (\zeta_l^2 + \eta_l^2)} e^{\frac{ik}{2z_o} [(\zeta_l - \zeta_o)^2 + (\eta_l - \eta_o)^2]} e^{\frac{ik}{2z_i} [(x-\zeta_l)^2 + (y-\eta_l)^2]} \end{aligned} \quad (10.21)$$

10.5.1 FT reduction

Let's focus on simplifying the phase terms. First, we note that we can complete the square for the integral over ζ_l and η .

$$\begin{aligned} U_i(x, y, z_i) &= \frac{k^2}{(i2\pi)^2 z_o z_i} e^{ik(z_o+z_i)} \int \int_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) \\ &\quad \times \int d\eta_l e^{\frac{ik}{2} \left[\zeta_l^2 \left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right) + \zeta_l \left(-\frac{2\zeta_o}{z_o} - \frac{2x}{z_i} \right) + \left(\frac{\zeta_o^2}{z_o} + \frac{x^2}{z_i} \right) \right]} \\ &\quad \times \int d\zeta_l e^{\frac{ik}{2} \left[\eta_l^2 \left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right) + \eta_l \left(-\frac{2\eta_o}{z_o} - \frac{2y}{z_i} \right) + \left(\frac{\eta_o^2}{z_o} + \frac{y^2}{z_i} \right) \right]} \\ &= \frac{k^2}{(i2\pi)^2 z_o z_i} e^{ik(z_o+z_i)} \int \int_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) \\ &\quad \times e^{\frac{ik}{2} \left(\frac{\zeta_o^2}{z_o} + \frac{x^2}{z_i} \right)} e^{\frac{-ik}{2} \left[\frac{\left(\frac{\zeta_o}{z_o} + \frac{x}{z_i} \right)^2}{-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o}} \right]} \int d\zeta_l e^{\frac{ik}{2} \left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right) \left[\zeta_l^2 - 2\eta_l \frac{\left(\frac{\zeta_o}{z_o} + \frac{x}{z_i} \right)}{\left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right)} + \left(\frac{\zeta_o}{z_o} + \frac{x}{z_i} \right)^2 \right]} \\ &\quad \times e^{\frac{ik}{2} \left(\frac{\eta_o^2}{z_o} + \frac{y^2}{z_i} \right)} e^{\frac{-ik}{2} \left[\frac{\left(\frac{\eta_o}{z_o} + \frac{y}{z_i} \right)^2}{-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o}} \right]} \int d\eta_l e^{\frac{ik}{2} \left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right) \left[\eta_l^2 - 2\eta_l \frac{\left(\frac{\eta_o}{z_o} + \frac{y}{z_i} \right)}{\left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right)} + \left(\frac{\eta_o}{z_o} + \frac{y}{z_i} \right)^2 \right]} \\ &= \frac{k^2}{(i2\pi)^2 z_o z_i} e^{ik(z_o+z_i)} \int \int_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -z_o) \\ &\quad \times \frac{2\pi}{-ik \left(-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o} \right)} e^{\frac{-ik}{2} \left[\frac{\left(\frac{\zeta_o}{z_o} + \frac{x}{z_i} \right)^2}{-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o}} - \left(\frac{\zeta_o^2}{z_o} + \frac{x^2}{z_i} \right) \right]} e^{\frac{-ik}{2} \left[\frac{\left(\frac{\eta_o}{z_o} + \frac{y}{z_i} \right)^2}{-\frac{1}{f} + \frac{1}{z_i} + \frac{1}{z_o}} - \left(\frac{\eta_o^2}{z_o} + \frac{y^2}{z_i} \right) \right]} \end{aligned} \quad (10.23)$$

where we completed the square to form a Gaussian integral. We wish to cancel out a quadratic phase, i.e., a term of the form $\exp[ik(x^2 + y^2)]$ which will act like a lens and distort the scalar field. This occurs by taking $z_o = z_i = f$. Then, for the special case of $z_o = z_i = f$, the Fresnel propagation from object to lens to image becomes

$$\begin{aligned} U_i(x, y, f) &= \frac{k^2}{(i2\pi)^2 f^2} e^{i2kf} \iint_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -f) \frac{2\pi f}{-ik} e^{-ik\frac{\zeta_o}{f}x} e^{-ik\frac{\eta_o}{f}y} \quad (10.24) \\ &= \frac{k}{i2\pi f} e^{i2kf} \iint_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -f) e^{-i\frac{k}{f}(\zeta_o x + \eta_o y)} \end{aligned}$$

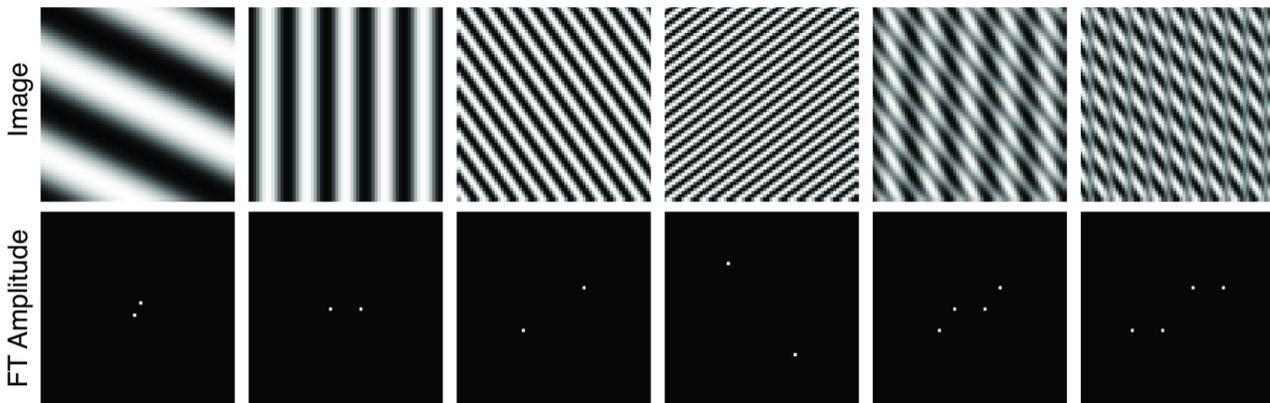
which is exactly proportional to the Fourier transform of the field at the object, with kx/f and ky/f in the role of spatial frequency. Note that the result is dimensionless, as expected.

A practical issue is that sometimes the alignment of optics does not satisfy, $z_o = z_i = f$, but rather $z_o = f$ and $z_i = f + \Delta$, where Δ is small. This leads to

$$U_i(x, y, f + \Delta) = \frac{k}{i2\pi f} e^{i2kf} \iint_{\Sigma_o} d\zeta_o d\eta_o U_o(\zeta_o, \eta_o, -f) e^{-i\frac{k}{f}(\zeta_o x + \eta_o y)} e^{-i\frac{k}{f^2}(\zeta_o^2 + \eta_o^2)} \quad (10.25)$$

which leads to a distortion equivalent to inserting a lens of focal length f^2/Δ at the Fourier plane. The distortion is small so long as $\Delta \ll f$.

Figure 3: Fourier transform pairs



The sequential placement of two such lens systems, the "4f" optical telescope, effectively computes the Fourier transform of the object with the first lens and the inverse Fourier transform with the second. The 4f telescope allows for spatial filtering of the object plane by manipulating the field in the Fourier plane. We also saw, using ABCD matrices, that the "4f" configuration teleports a ray at the object plane to a ray at the image plane. This is useful; you can move an image and spatially filter it at the intermediate, or so-called conjugate, plane.